MATRICES (PAPER 1 GROUP B)- FIRST YEAR Lecture 03

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1 Rank of a Matrix:

Let A be a matrix. A natural number r is said to be rank of A if

- there is at least one $r-th$ order non-singular square sub-matrix of A.
- every square sub-matrix of A of order greater than r is singular.

Illustration: (i) Find the rank of the matrix $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ using definition.

Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. We see all the 3rd order sub matrices of A are $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

(there will be $\binom{4}{3} = 4$ number of such sub-matrices) which are all singular as all the elements in the 3rd row are 0.

Then we go for 2nd order sub matrix. We see through the sub matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is singular, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is non singular because $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$. So by definition, Rank of $A = 2$.

Note:

- (i) For an nth order square matrix A, if $detA \neq 0$, rank of $A = n$.
- (ii) The rank of null matrix is 0.
- (iii) Rank of nth order identity matrix is n.
- (iv) Rank can be found by transforming the matrix to Echelon Matrix.

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Illustration Find the rank of the rectangular matrix $\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$

Let $A = \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix} \longrightarrow_{R_2-2R_1} \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix} \longrightarrow_{R_4-3R_1}$
$\begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{bmatrix} \longrightarrow_{R_3+R_2} \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{bmatrix} \longrightarrow_{R_{34}} \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Since there exist three non zero rows, hence the rank of the matrix is 3.

2 System of linear equations

A set of m linear equations in n number of unknowns is known as a system of linear equations. The general form of this system is shown below:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n &= b_2 \\ .... &= ... \\ a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n &= b_m \end{aligned}$$

where $x_1, x_2, x_3, ..., x_n$ are unknowns.

2.1 Homogeneous and Non-homogeneous system of linear equations

If all the constant terms $b_1, b_2, b_3, ...b_m$ of (1) are 0 then the system is called homogeneous system.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n &= 0 \\ .... &= ... \\ a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n &= 0 \end{aligned}$$

If at least one of $b_1, b_2, ..., b_m$ is zero then the system is called non-homogeneous system.

2.2 Trivial solution of a Homogeneous system

In a homogeneous system of m equations in n unknowns $x_1, x_2, ..., x_n$ we see $x_1 = 0, x_2 = 0, ..., x_n = 0$ must satisfy all the equations of the system. This solution $(0, 0, ..., 0)$ is called the trivial solution.

2.3 Consistent and Inconsistent system of linear equations

A system of equations is said to be consistent if the system has at least one solution.

A system of equations is said to be inconsistent if the system has no solution.

Theorem 1. On consistency of non-homogeneous system *Let*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n &= 0 \\ .... &= ... \\ a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n &= 0 \end{aligned}$$

be a system of m number of linear equations in n number of unknowns $x_1, x_2, x_3, ..., x_n$. Let us consider the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ ... & ... & ... & ... & \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{pmatrix}$$

I. *The system is consistent if and only if Rank of A=Rank of B.*

II. *The consistent system has unique solution if and only if Rank of A=number of unknowns*

III. *The consistent system has many solution if Rank of A $<$ number of unknowns.*

3 Hermitian and Skew Hermitian matrices

A complex $n \times n$ matrix A is said to be Hermitian if $A^0 = A$ and skew Hermitian if $A^0 = -A$, where $A^0 = \bar{A}^t$.

Therefore a complex $n \times n$ matrix $A = (a_{ij})$ is Hermitian if $\bar{a}_{ij} = a_{ji}$ and skew Hermitian if $\bar{a}_{ij} = -a_{ij}$ for $i = 1, 2, ..., n;\ j = 1, 2, 3..., n$.

Example of a Hermitian matrix are

$$\begin{pmatrix} 1 & 2+i & -1 \\ 2-i & 2 & i \\ -1 & -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3i \\ 2 & 0 & 4 \\ -3i & 4 & 2 \end{pmatrix}.$$

Example of skew Hermitian matrix are

$$\begin{pmatrix} 0 & 2 & i \\ -2 & 0 & 1+i \\ -i & -1+i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2i & 1+2i \\ 2i & i & 1 \\ -1+2i & -1 & 0 \end{pmatrix}.$$

In particular, a real Hermitian matrix is a real symmetric matrix and a real skew Hermitian matrix is a real skew symmetric matrix.

Theorem 2. *If $H = P + iQ$ be a Hermitian matrix, then*

1. *(i) the diagonal elements of H are real numbers.*
2. *(ii) P is a real symmetric matrix and Q is a real skew symmetric matrix.*

Proof. (i) Let $H = (h_{ij})$. Then $\bar{h}_{ij} = h_{ji}$.
So $\bar{h}_{ij} = h_{ji}$ and this proves that h_{ij} is purely real.
(ii) Since H is Hermitian matrix, then $\bar{H}^t = H$.
Therefore, $P^t - iQ^t = P + iQ$. This implies that $P^t = P$ and $Q^t = -Q$ proving that P is symmetric and Q is skew symmetric. □

Theorem 3. *A complex square matrix can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.*

Proof. Let A be a square matrix and A can be represented as sum of B and C, two square matrix with same dimension with A. $A = B + C$.
Now A can be represented as $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$, where $B = \frac{1}{2}(A + A^\theta)$ and $C = \frac{1}{2}(A - A^\theta)$. Here $A^\theta = \bar{A}^t$.
We have to show that B is Hermitian and C is skew Hermitian matrix.
$B^\theta = \frac{1}{2}(A + A^\theta)^\theta = \frac{1}{2}(A^\theta + (A^\theta)^\theta) = \frac{1}{2}(A^\theta + A) = B$.
Also $C^\theta = \frac{1}{2}(A - A^\theta)^\theta = \frac{1}{2}(A^\theta - (A^\theta)^\theta) = \frac{1}{2}(A^\theta - A) = -C$.
Hence B is Hermitian matrix and C is skew Hermitian matrix, which completes the proof. □

4 Unitary matrices

A complex matrix $n \times n$ matrix A is said to be unitary if $AA^\theta = I_n$.

Example: $\begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$, $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$

Note: For a real unitary A, $AA^\theta = A^\theta A = I_n$. Hence every real unitary matrix is real orthogonal matrix.