Lecture : 04

B. Sc. (Hon.) Part-I

Paper - II

Tangent & Normal

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I. DERIVATIVE OF ARC LENGTH (CARTESIAN)

Let $P(x, y)$ be the given point, and $Q(x + \Delta x, y + \Delta y)$ be any point near P on the curve.

Let s denote the length of the arc AP measured from a fixed point A on the curve, and let $s + \Delta s$ denote the are AQ, so that arc $PQ = \Delta s$. Here, s is a function of x and hence y. We shall assume the fundamental limit

$$\lim_{p \to Q} \frac{chordPQ}{arcPQ} = 1$$

From the figure, $(chord\ PQ)^2 = PR^2 + QR^2 = (\Delta x)^2 + (\Delta y)^2$

$$\left(\frac{chordPQ}{\Delta}\right)^2 . \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

Now let $Q \to P$ as a limiting position; then $\Delta \to 0$ an we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{1}$$

Also we get

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \tag{2}$$

II. VALUE OF SIN ϕ AND COS ϕ

From ΔPQR, $\sin QPR = \frac{RQ}{PQ} = \frac{\Delta y}{\Delta s} . \frac{\Delta s}{PQ}$.

In the limiting position when $Q \to P$, the secant PQ becomes the tangent at P, $\angle QPR \to \phi$ and $\Delta s \to 0$ and

$\frac{\Delta s}{PQ} = \frac{arcPQ}{chordPQ} \to 1.$

$$\sin\phi = \lim_{\Delta\to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \tag{3}$$

Similarly,

$$\cos\phi = \lim_{\Delta s\to 0}\frac{\Delta x}{\Delta s} = \frac{dx}{ds} \tag{4}$$

From the above we get the relation

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \tag{5}$$

Cor.1 If $x = \phi(t),\ y = \psi(t)$ then we get the relation

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2$$

III. ILLUSTRATION

Example 1. Find the equation of the tangent at (x, y) to the curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$

Here the equation of the curve is $f(x, y) \equiv (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} - 1 = 0$.

The equation of the tangent is

$$\begin{aligned} (X-x)f_x + (Y-y)f_y &= 0 \\ i.e.\ (X-x).\frac{2}{3}x^{-\frac{1}{3}}/a^{\frac{2}{3}} + (Y-y).\frac{2}{3}y^{-\frac{1}{3}}/b^{\frac{2}{3}} &= 0 \\ i.e., Xx^{-\frac{1}{3}}/a^{\frac{2}{3}} + Yy^{-\frac{1}{3}}/b^{\frac{2}{3}} &= (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} \\ Xx^{-\frac{1}{3}}/a^{\frac{2}{3}} + Yy^{-\frac{1}{3}}/b^{\frac{2}{3}} &= 1. \end{aligned}$$

Example 2. Find the condition that the conics

$$ax^2 + by^2 = 1\ and\ a_1x^2 + b_1y^2 = 1$$

shall cut orthogonally.

The equations of the conics are

$$
\begin{aligned}
f(x,y) &\equiv ax^2 + by^2 - 1 = 0 \\
g(x,y) &\equiv a_1x^2 + by^2 - 1 = 0
\end{aligned}
$$

The condition for orthogonallay at (x, y)

$$
\begin{aligned}
f_x g_x + f_y g_y &= 0, \\
2ax.2a_1x + 2by.2b_1y &= 0 \\
aa_1x^2 + bb_1y^2 &= 0.
\end{aligned}
$$

Substituting above equations we get

$$(a - a_1)x^2 + (b - b_1)y^2 = 0$$

Comparing we get,

$$\frac{a - a_1}{aa_1} = \frac{b - b_1}{bb_1},\ or,\ \frac{1}{a_1} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{b}$$

which is the required conditions.