Lecture #: 04

Physics Course: Methods of Mathematical Physics

For B. Sc. III

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1 Tensor Notations

The velocity of the wind at the top of Eiffel's tower, at a given moment, can be represented by a *vector* $\mathbf{v}$ with components, in some local, given, basis, $\{v^i\}$ $(i = 1, 2, 3)$. The velocity of the wind is defined at any point $\mathbf{x}$ of the atmosphere at any time t : we have a *vector field* $v^i(\mathbf{x}, t)$.

The water's temperature at some point in the ocean, at a given moment, can be represented by a *scalar* T . The field $T(\mathbf{x}, t)$ is a *scalar field*.

The state of stress at a given point of the Earth's crust, at a given moment, is represented by a *second order tensor* $\boldsymbol{\sigma}$ with components $\{\sigma^{ij}\}$ $(i = 1, 2, 3;\ j = 1, 2, 3)$. In a general model of continuous media, where it is not assumed that the stress tensor is symmetric, this means that we need 9 scalar quantities to characterize the state of stress. In more particular models, the stress tensor is symmetric, $\sigma^{ij} = \sigma^{ji}$, and only six scalar quantities are needed. The stress field $\sigma^{ij}(\mathbf{x}, t)$ is a *second order tensor field*.

Tensor fields can be combined, to give other fields. For instance, if n_i is a unit vector considered at a point inside a medium, the vector

$$\tau^i(\mathbf{x}, t) \;=\; \sum_{j=1}^{3} \sigma^{ij}(\mathbf{x}, t)\, n_j(\mathbf{x}) \;=\; \sigma^{ij}(\mathbf{x}, t)\, n_j(\mathbf{x}) \qquad (1)$$

represents the traction that the medium at one side of the surface defined by the normal n_i exerts the medium at the other side, at the considered point.

As a further example, if the deformations of an elastic solid are small enough, the stress tensor is related linearly to the strain tensor (Hooke's law). A linear relation between two second order tensors means that each component of one tensor can be computed as a linear combination of all the components of the other tensor:

$$\sigma^{ij}(\mathbf{x}, t) \;=\; \sum_{k=1}^{3} \sum_{\ell=1}^{3} c^{ijk\ell}(\mathbf{x})\, \varepsilon_{k\ell}(\mathbf{x}, t) \;=\; c^{ijk\ell}(\mathbf{x})\, \varepsilon_{k\ell}(\mathbf{x}, t)\, . \qquad (2)$$

The *fourth order tensor* c^{ijkl} represents a property of an elastic medium: its elastic stiffness. As each index takes 3 values, there are $3 \times 3 \times 3 \times 3 = 81$ scalars to define the elastic stiffness of a solid at a point (assuming some symmetries we may reduce this number to 21, and asuming isotropy of the medium, to 2).

We are not yet interested in the physical meaning of the equations above, but in their structure. First, tensor notations are such that they are independent on the coordinates being used. This is not obvious, as changing the coordinates implies changing the local basis where the components of vectors and tensors are expressed. That the two equalities equalities above hold for any coordinate system, means that all the components of all tensors will change if we change the coordinate system being used (for instance, from Cartesian to spherical coordinates), but still the two sides of the expression will take equal values.

The mechanics of the notation, once understood, are such that it is only possible to write expressions that make sense (see a list of rules at the end of this section).

For reasons about to be discussed, indices may come in upper or lower positions, like in v^i , f_i or $T_i{}^j$. The definitions will be such that in all tensor expression (i.e., in all expressions that will be valid for all coordinate systems), the sums over indices will always concern an index in lower position an one index on upper position. For instance, we may encounter expressions like

$$\varphi \;=\; \sum_{i=1}^{3} A_i B^i \;=\; A_i B^i \tag{3}$$

or

$$A_i \;=\; \sum_{j=1}^{3} \sum_{k=1}^{3} D_{ijk} E^{jk} \;=\; D_{ijk} E^{jk} \quad . \tag{4}$$

These two equations (as equations 1 and 2) have been written in two versions, one with the sums over the indices explicitly indicated, and another where this sum is implicitly assumed. This implicit notation is useful as one easily forgets that one is dealing with sums, and that it happens that, with respect to the usual tensor operations (sum with another tensor field, multiplication with another tensor field, and derivation), a sum of such terms is handled as one single term of the sum could be handled.

In an expression like $A_i = D_{ijk} E^{jk}$ it is said that the indices j and k have been *contracted* (or are "dummy indices"), while the index i is a *free index*. A tensor equation is assumed to hold for all possible values of the free indices.

In some spaces, like our physical 3-D space, it is posible to define the distance between two points, and in such a way that, in a local system of coordinates, approximately Cartesian, the distance has approximately the Euclidean form (square root of a sum of squares). These spaces are called *metric spaces*. A mathematically convenient manner to introduce a metric is by defining the length of an arc Γ by $S = \int_\Gamma ds$, where, for instance, in Cartesian coordinates, $ds^2 = dx^2 + dy^2 + dz^2$ or, in spherical coordinates, $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$. In general, we write $ds^2 = g_{ij} dx^i dx^j$, and we call $g_{ij}(\mathbf{x})$ the *metric field* or, simply, the *metric*.

The components of a vector $\mathbf{v}$ are associated to a given basis (the vector will have different components on different basis). If a basis $\mathbf{e}_i$ is given, then, the components v^i are defined through $\mathbf{v} = v^i \mathbf{e}_i$ (implicit sum). The *dual basis* of the basis $\{\mathbf{e}_i\}$ is denoted $\{\mathbf{e}^i\}$ and is defined by the equation $\mathbf{e}_i \mathbf{e}^j = \delta_i{}^j$ (equal to 1 if i are the same index and to 0 if not). When there is a metric, this equation can be interpreted as a scalar vector product, and the dual basis is just another basis (identical to the first one when working with Cartesian coordinates in Euclidena spaces, but different in general). The properties of the dual basis will be analyzed later in the chapter. Here we just need to recall that if v^i are the components of the vector $\mathbf{v}$ on the basis $\{\mathbf{e}_i\}$ (remember the expression $\mathbf{v} = v^i \mathbf{e}_i$), we will denote by v_i are the components of the vector $\mathbf{v}$ on the basis $\{\mathbf{e}^i\}$: $\mathbf{v} = v_i \mathbf{e}^i$. In that case (metric spaces) the components on the two basis are related by $v_i = g_{ij} v^i$: It is said that "the metric tensor ascends (or descends) the indices".

Here is a list with some rules helping to recognize tensor equations:

- A tensor expression must have the same *free* indices, at the top and at the bottom, of the two sides of an equality. For instance, the expressions

$$\begin{aligned} \varphi &= A_i \, B^i \\ \varphi &= g_{ij} \, B^i \, C^j \\ A_i &= D_{ijk} \, E^{jk} \\ D_{ijk} &= \nabla_i F_{jk} \end{aligned} \tag{5}$$

  are valid, but the expressions

$$\begin{aligned} A_i &= F_{ij} \, B^i \\ B^i &= A_j \, C^j \\ A_i &= B^i \end{aligned} \tag{6}$$

  are not.

- Sum and multiplication of tensors (with eventual "contraction" of indices) gives tensors. For instance, if D_{ijk} , G_{ijk} and $H_i{}^j$ are tensors, then

$$\begin{aligned} J_{ijk} &= D_{ijk} + G_{ijk} \\ K_{ijk\ell}{}^m &= D_{ijk} \, H_\ell{}^m \\ L_{ik\ell} &= D_{ijk} \, H_\ell{}^j \end{aligned} \tag{7}$$

  also are tensors.

- True (or "covariant") derivatives of tensor felds give tensor fields. For instance, if E^{ij} is a tensor field, then

$$\begin{aligned} M_i{}^{jk} &= \nabla_i E^{jk} \\ B^j &= \nabla_i E^{ij} \end{aligned} \tag{8}$$

  also are tensor fields. But partial derivatives of tensors do not define, in general, tensors. For instance, if E^{ij} is a tensor field, then

$$\begin{aligned} M_i{}^{jk} &= \partial_i V^{jk} \\ B^j &= \partial_i V^{ij} \end{aligned} \tag{9}$$

  are not tensors, in general.