Lecture #: 02

Physics Course: Methods of Mathematical Physics

For B. Sc. III

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Divergence and Curl

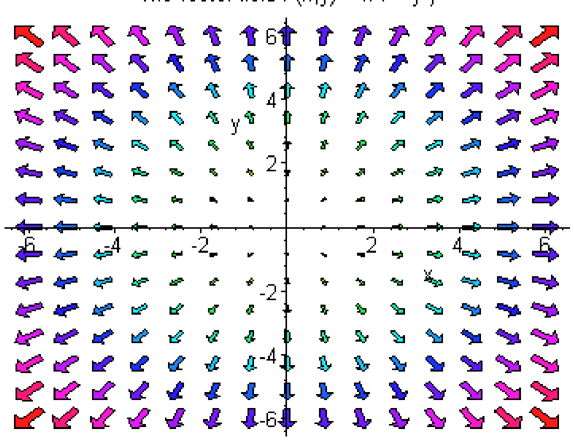
"Del", ∇ - A defined operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

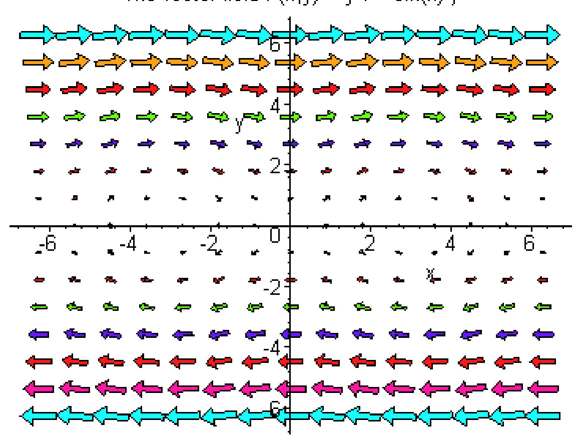
The **gradient** of a function (at a point) is a vector that points in the direction in which the function increases most rapidly.

A **vector field** is a vector function that can be thought of as a velocity field of a fluid. At each point it assigns a vector that represents the velocity of a particle at that point.

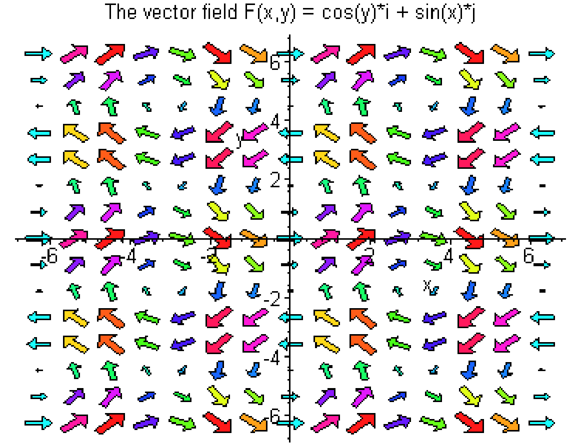
The vector field F(x,y) = x*i + y*j



The vector field F(x,y) = y*i + sin(x)*j



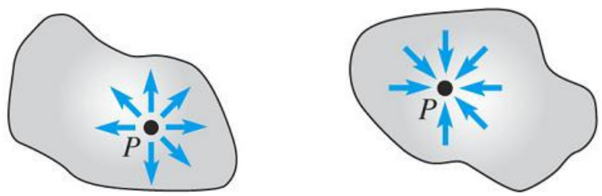
The vector field F(x,y) = cos(y)*i + sin(x)*j



The **flux** of a vector field is the volume of fluid flowing through an element of surface area per unit time.

The **divergence** of a vector field is the flux per unit volume.

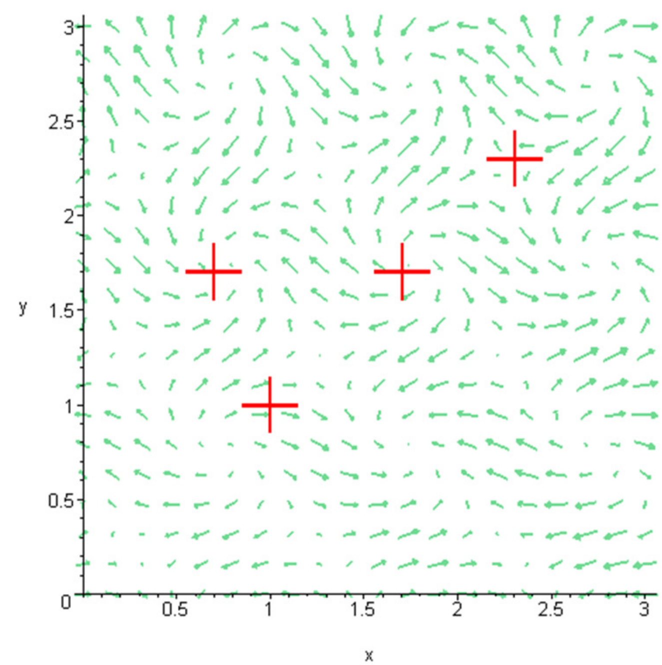
The divergence of a vector field is a number that can be thought of as a measure of the rate of change of the density of the fluid at a point.



(a) div $\mathbf{F}(P) > 0$; P a source (b) div $\mathbf{F}(P) < 0$; P a sink

The **curl** of a vector field measures the tendency of the vector field to rotate about a point.

The curl of a vector field at a point is a vector that points in the direction of the axis of rotation and has magnitude represents the speed of the rotation.



Vector Field

$$\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

Scalar Function

$$f(x, y, z)$$

Gradient $grad(f)$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Divergence $div(\mathbf{F})$

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = P_x + Q_y + R_z$$

Curl $curl(\mathbf{F})$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

$$\nabla \times \mathbf{F} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$$

$$\mathbf{F}(x,y,z)=\left\langle xe^{-z},4yz^2,3ye^{-z}\right\rangle$$

$$div(\mathbf{F})=\nabla\cdot\mathbf{F}=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\cdot\left\langle xe^{-z},4yz^2,3ye^{-z}\right\rangle=\boxed{e^{-z}+4z^2-3ye^{-z}}$$

$$curl(\mathbf{F})=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xe^{-z} & 4yz^2 & 3ye^{-z}\end{vmatrix}=\left\langle 3e^{-z}-8yz^2,-\left(0-\left(-xe^{-z}\right)\right),\ \ 0\ \ \right\rangle$$

$$\boxed{\left\langle 3e^{-z}-8yz^2,-xe^{-z},0\right\rangle}$$

$$grad\,(scalar\ function) = \textbf{Vector Field}$$

$$div\,(\textbf{Vector Field}) = scalar\ function$$

$$curl\,(\textbf{Vector Field}) = \textbf{Vector Field}$$

Which of the 9 ways to combine grad, div and curl by taking one of each. Which of these combinations make sense?

~~$grad\,(grad\,(f))$~~ **Vector Field**	✓ $div\,(grad\,(f))$ **Vector Field**	✓ $curl\,(grad\,(f))$ **Vector Field** $\boxed{\mathbf{0}\ \text{vector}}$
✓ $grad\,(div\,(\mathbf{F}))$ *scalar function*	~~$div\,(div\,(\mathbf{F}))$~~ *scalar function*	~~$curl\,(div\,(\mathbf{F}))$~~ *scalar function*
~~$grad\,(curl\,(\mathbf{F}))$~~ **Vector Field**	✓ $div\,(curl\,(\mathbf{F}))$ **Vector Field** $\boxed{0\ \text{scalar}}$	✓ $curl\,(curl\,(\mathbf{F}))$ **Vector Field**

2 of the above are always zero.

Verify the given identity. Assume continuity of all partial derivatives.

$curl\left(grad\left(f\right)\right)=\mathbf{0}.$

$$grad\left(f\right)=\left\langle f_x, f_y, f_z\right\rangle$$

$$curl\left(grad\left(f\right)\right)=\nabla\times\nabla f=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f_x & f_y & f_z\end{vmatrix}$$

$$\nabla\times\nabla f=\left\langle f_{zy}-f_{yz},-\left(f_{zx}-f_{xz}\right),f_{yx}-f_{xy}\right\rangle=\mathbf{0}$$

since mixed partial derivatives are equal.

Verify the given identity. Assume continuity of all partial derivatives.

$div\left(curl\left(\mathbf{F}\right)\right)=0.$

Let $\mathbf{F}=\left\langle P\left(x,y,z\right),Q\left(x,y,z\right),R\left(x,y,z\right)\right\rangle$

$$curl\left(\mathbf{F}\right)=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$$curl\left(\mathbf{F}\right)=\left\langle R_y-Q_z, P_z-R_x, Q_x-P_y\right\rangle$$

$$div\left(curl\left(\mathbf{F}\right)\right)=\left(R_y-Q_z\right)_x+\left(P_z-R_x\right)_y+\left(Q_x-P_y\right)_z$$

$$div\left(curl\left(\mathbf{F}\right)\right)=R_{yx}-Q_{zx}+P_{zy}-R_{xy}+Q_{xz}-P_{yz}$$

$$div\left(curl\left(\mathbf{F}\right)\right)=\cancel{R_{yx}}-\cancel{Q_{zx}}+\cancel{P_{zy}}-\cancel{R_{xy}}+\cancel{Q_{xz}}-\cancel{P_{yz}}=0$$