Lecture : 13

B. Sc. (Hon.) Part-I

Paper - I

Physics Course: Waves and Vibration

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I. FOURIER TRANSFORMATION

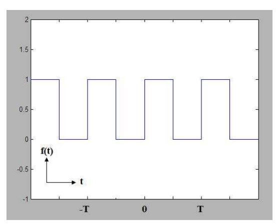
Fourier transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. The Fourier Transform shows that any waveform can be re-written as the sum of sinusoidal functions.

A. Fourier Series

The Fourier series breaks down a periodic function into the sum of sinusoidal functions. It is the Fourier Transform for periodic functions. To start the analysis of Fourier Series, let's define periodic functions. A function is periodic, with fundamental period T, if the following is true for all t:

$$f(t+T) = f(t) \tag{1}$$

For example



The square waveform of Figure 1 has a fundamental period of T.

II. THE DIRICHLET CONDITIONS

Not all functions can be expanded in Fourier series. They are transformable if they fulfil certain conditions, known as the Dirichlet conditions.

- The function must be periodic.
- It must be single-valued and continuous, except possibly at a finite number of finite discontinuities ('piece-wise continuous').
- It must have only a finite number of maxima and minima within one period
- The integral over one period of $|f(t)|$ must converge.

Now, since all functions $f(t)$ may be written as the sum of an odd and an even part,

$$\begin{aligned} f(t) &= \frac{1}{2}[f(t) + f(-t)] + \frac{1}{2}[f(t) - f(-t)], \\ &= f_{\text{even}}(t) + f_{\text{odd}}(t) \end{aligned} \tag{2}$$

we can write any function as the sum of a sine series and a cosine series.

Let's define a Fourier Series now. A Fourier Series, with period T, is an infinite sum of sinusoidal functions (cosine and sine), each with a frequency that is an integer multiple of $1/T$ (the inverse of the fundamental period). The Fourier Series also includes a constant, and hence can be written as:

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{2\pi mt}{T}\right) + b_m \sin\left(\frac{2\pi mt}{T}\right) \right] \tag{3}$$

The constants a_0, a_m, b_m are the coefficients of the Fourier Series. The question now is:

All the terms of a Fourier series are mutually orthogonal, i.e., the integrals, over one period T, of the product of any two terms have the following properties:

$$\int_0^T \sin\left(\frac{2\pi mt}{T}\right) \cos\left(\frac{2\pi nt}{T}\right) dt = 0, \text{ for all } n \;\&\; m,$$

$$\begin{aligned} \int_0^T \cos\left(\frac{2\pi mt}{T}\right) \cos\left(\frac{2\pi nt}{T}\right) dt = \quad & T \text{ for } m = n = 0, \\ & \frac{T}{2} \text{ for } m = n > 0, \\ & 0 \text{ for } m \neq n \end{aligned}$$

$$\int_0^T \sin\left(\frac{2\pi mt}{T}\right)\sin\left(\frac{2\pi nt}{T}\right)dt = \begin{cases} 0 \text{ for } m = n = 0, \\ \frac{T}{2} \text{ for } m = n > 0, \\ 0 \text{ for } m \neq n \end{cases}$$

where m and n are integers greater than or equal to zero.

A. Fourier coefficients

Multiplying (3) by $\cos\left(\frac{2\pi nt}{T}\right)$, integrating over one full period T in t and changing the order of the summation and integration, we get

$$\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)dt = \int_0^T \frac{a_0}{2}\cos\left(\frac{2\pi nt}{T}\right)dt + \sum_{m=1}^{\infty}\int_0^T \left[a_m\cos\left(\frac{2\pi mt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right) + b_m\sin\left(\frac{2\pi mt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)\right]dt \tag{4}$$

Using above orthogonality conditions, we find that when $n = 0$ (4) becomes

$$\int_0^T f(t)dt = \frac{a_0}{2}T \tag{5}$$

This implies

$$\boxed{a_0 = \frac{2}{T}\int_0^T f(t)dt}$$

When $n \neq 0$ the only non-vanishing term on the RHS of (4) occurs when $m = n$, and so

$$\int_0^T f(t)\cos\left(\frac{2\pi mt}{T}\right)dt = \frac{a_m}{2}T \tag{6}$$

This implies

$$\boxed{a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi mt}{T}\right)dt}$$

The other Fourier coefficients b_m may be found by repeating the above process but multiplying by $\sin\left(\frac{2\pi nt}{T}\right)$ instead of $\cos\left(\frac{2\pi nt}{T}\right)$, i.e.,

$$\boxed{b_m = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi mt}{T}\right)dt}$$

III. COMPLEX FOURIER SERIES

As a Fourier series expansion in general contains both sine and cosine parts, it may be written more compactly using a complex exponential expansion. This simplification makes use of the property that $e^{imt} = \cos mt + i\sin mt$. The complex Fourier series expansion is written

$$f(t) = \sum_{-\infty}^{\infty} c_m \exp\left(\frac{2\pi imt}{T}\right) \tag{7}$$

where the Fourier coefficients are given by

$$c_m = \frac{1}{T}\int_0^T f(t)\exp\left(-\frac{2\pi imt}{T}\right)dt \tag{8}$$

This relation can be derived, in a similar manner to that of section 12.2, by multiplying (12.9) by $\exp\left(-\frac{2\pi int}{T}\right)$ before integrating and using the orthogonality relation.

The complex Fourier coefficients in (8) have the following relations to the real Fourier coefficients:

$$\begin{aligned} c_m &= \frac{1}{2}(a_m - ib_m) \\ c_{-m} &= \frac{1}{2}(a_m + ib_m) \\ c_0 &= \frac{1}{2}a_0 \end{aligned} \tag{9}$$