Lecture #: 02

Physics Course: Relativity

For B. Sc. (Hon.) and B. Sc. (IT)

Dr. Sudhaker Upadhyay

Assistant professor

Department of Physics

K.L.S. College, Nawada

I. NON-INERTIAL FRAME OF REFERENCE

Now we will discuss what Newton's equations of motion look like in non-inertial frames. There are many ways in which a reference frame can be non-inertial. Here we will just consider one type: reference frames that rotate.

Let's start with the inertial frame S drawn in the figure with coordinate axes x, y and z. Our goal is to understand the motion of particles as seen in a non-inertial frame S' with axes x', y' and z', which is rotating with respect to S. Consider a particle that is sitting stationary in the S' frame. Then, from the perspective of frame S it will appear to be moving with velocity

$$\dot{r} = \omega \times r. \tag{1}$$

ω is the angular velocity. In the present case $\omega = \dot{\theta}\hat{z}$

Let $e'_i, i = 1; 2; 3$ be the unit vectors that point along the x', y' and z' directions of S'. Then these also rotate with velocity

$$\dot{e}'_i = \omega \times e'_i. \tag{2}$$

II. VELOCITY AND ACCELERATION IN A ROTATING FRAME

Consider now a particle which is no longer stuck in the S' frame, but moves on some trajectory. We can measure the position of the particle in the inertial frame S, where, using the summation convention, we write

$$\vec{r} = r_i e_i$$

Here the unit vectors e_i, with $i = 1; 2; 3$ point along the axes of S. Alternatively, we can measure the position of the particle in frame S', where the position is

$$\vec{r} = r'_i e'_i$$

Note that the position vector $\vec{r}$ is the same in both of these expressions: but the coordinates r_i and r'_i differ because they are measured with respect to different axes. Now, we can compute an expression for the velocity of the particle. In frame S, it is simply

$$\dot{\vec{r}} = \dot{r}_i e_i$$

because the axes e_i do not change with time. However, in the rotating frame S', the velocity of the particle is

$$\begin{aligned} \dot{\vec{r}} &= \dot{r}'_i e'_i + r'_i \dot{e}'_i \\ &= \dot{r}'_i e'_i + r'_i \omega \times e'_i \\ &= \dot{r}'_i e'_i + \omega \times \vec{r} \end{aligned} \qquad (3)$$

We'll introduce a slightly novel notation to help highlight the physics hiding in these two equations. We write the velocity of the particle as seen by an observer in frame S as

$$\left(\frac{d\vec{r}}{dt}\right)_S = \dot{r}_i e_i$$

Similarly, the velocity as seen by an observer in frame S' is just

$$\left(\frac{d\vec{r}}{dt}\right)_{S'} = \dot{r}'_i e'_i$$

From (3)

$$\left(\frac{d\vec{r}}{dt}\right)_S = \left(\frac{d\vec{r}}{dt}\right)_{S'} + \omega \times \vec{r}$$

This is not completely surprising: the difference is just the relative velocity of the two frames.

What about acceleration? We can play the same game. In frame S, we have

$$\ddot{\vec{r}} = \ddot{r}_i e_i$$

while in frame S', the expression is a little more complicated. Differentiating (3) once more, we have

$$\begin{aligned} \ddot{\vec{r}} &= \ddot{r}'_i e'_i + \dot{r}'_i \dot{e}'_i + \dot{r}'_i \omega \times e'_i + r'_i \dot{\omega} \times e'_i + r'_i \omega \times \dot{e}'_i \\ &= \ddot{r}'_i e'_i + 2\dot{r}'_i \omega \times e'_i + \dot{\omega} \times \vec{r} + r'_i \omega \times (\omega \times e'_i) \end{aligned} \tag{4}$$

As with velocities, the acceleration seen by the observer in S is $\ddot{r}_i e_i$ while the acceleration seen by the observer in S' is $\ddot{r}'_i e'_i$. Equating the two equations above gives us

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_S = \left(\frac{d^2\vec{r}}{dt^2}\right)'_S + 2\omega \times \left(\frac{d\vec{r}}{dt}\right)_{S'} + \dot{\omega} \times \vec{r} + \omega \times (\omega \times \vec{r}) \tag{5}$$

This equation contains the key to understanding the motion of particles in a rotating frame.

III. NEWTON'S EQUATION OF MOTION IN A ROTATING FRAME

With the hard work behind us, let's see how a person sitting in the rotating frame S' would see Newton's law of motion. We know that in the inertial frame S, we have

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_S = F \tag{6}$$

Using (5), in frame S', we have

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S'} = F - 2m\omega \times \left(\frac{d\vec{r}}{dt}\right)_{S'} - m\dot{\omega} \times \vec{r} - m\omega \times (\omega \times \vec{r}) \tag{7}$$

In other words, to explain the motion of a particle an observer in S' must invoke the existence of three further terms on the right-hand side of Newton's equation. These are called fictitious forces. Viewed from S', a free particle doesn't travel in a straight line and these fictitious forces are necessary to explain this departure from uniform motion.

In Above expression second is the Coriolis force; the third term is called the Euler force; the last term is called the centrifugal force.