Lecture : 06

# Physics Course: Methods of Mathematical Physics

For B. Sc. (Hon.) III

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## I. CAUCHY-RIEMANN EQUATIONS

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that $u$ and $v$ satisfy the CauchyRiemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

If the partial derivatives in above equation are continuous in $\mathcal{R}$, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$.

The functions $u(x, y)$ and $v(x, y)$ are sometimes called conjugate functions. If the second partial derivatives of $u$ and $v$ with respect to $x$ and $y$ exist and are continuous in a region $\mathcal{R}$, then they satisfy Laplace's equation.

Functions such as $u(x, y)$ and $v(x, y)$ which satisfy Laplace's equation in a region $\mathcal{R}$ are called harmonic functions

## II. ZERO

A zero of a given function f is a number $z_0$ such that $f(z_0) = 0$. It is possible that a function of a real variable can have more zeros when the domain of definition is enlarged.

## III. SINGULAR POINTS

A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$. There are various types of singularities:

### A. Isolated singularities

The point $z_0$ is called an isolated singularity or isolated singular point of $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than $z_0$.

### B. Poles

If $z_0$ is an isolated singularity and we can find a positive integer $n$ such that $\lim_{z\to z_0}(z-z_0)^n f(z) \neq 0$, then $z = z_0$ is called a pole of order $n$. If $n = 1$, $z_0$ is called a simple pole.

### C. Essential singularities

An isolated singularity that is not a pole or removable singularity is called an essential singularity.

## IV. TAYLOR'S THEOREM

Let $f(z)$ be analytic inside and on a simple closed curve $C$. Then $f(z)$ can be expanded as a power series about $z = a$ as

$$f(z) = f(a) + (z-a)f'(a) + \frac{f''(a)}{2!}(z-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(z-a)^n + ... \tag{2}$$

This is called Taylor's theorem and the series is called a Taylor series or expansion for $f(z)$.

If the nearest singularity of $f(z)$ is at infinity, the radius of convergence is infinite, i.e., the series converges for all $z$.

If $a = 0$, the resulting series is often called a Maclaurin series.