SUCCESSIVE DIFFERENTIATION

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$f(x)$ is a function of x. The derivative of $f(x)$ is called first order derivative denoted as $f'(x)$. The derivative of $f'(x)$ is called second order derivative denoted as $f''(x)$. Similarly, the derivative of the n times of the function $f(x)$ is called *nthderivative* of $f(x)$ defined as $f^n(x)$.

If $y = f(x)$, the successive derivatives are also denoted as

$$\begin{array}{ccccc} y_1 & y_2 & y_3 & \cdots & y_n \\ or,\ y' & y'' & y''' & \ldots & y' \\ or,\ \dot{y} & \ddot{y} & \dddot{y} & \ldots & \ldots \\ f'(x) & f''(x) & f'''(x) & \ldots & f^n(x) \\ Df(x) & D^2f(x) & D^3f(x) & \ldots & D^nf(x). \end{array}$$

Here D standing for $\frac{d}{dx}$.

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The nth derivatives of some special functions

(i) $y = x^n$, where n is a positive integer. Then

$$y_1 = nx^{n-1}, \quad y_2 = n(n-1)x^{n-2}, \ldots$$

and proceeding in a similar manner,

$$\begin{aligned} y_r &= n(n-1)(n-2)\ldots\{n-(r-1)\}x^{n-r}(r<n) \\ \therefore y_n &= n(n-1)(n-2)\ldots\{n-(n-1)\}x^{n-n} = n! \\ i.e., D^n(x^n) &= n! \end{aligned}$$

(ii) $y = (ax + b)^m$, where m is any number,

$$y_1 = ma(ax+b)^{m-1}, \quad y_2 = m(m-1)a^2(ax+b)^{m-2}, ...$$

$$y_n = m(m-1)(m-2)...\{m-(r-1)\}a^n(ax+b)^{m-n}(n < m)$$
$$y_n = m(m-1)(m-2)...\{m-(n-1)\}a^n(ax+b)^{m-n}$$
$$i.e., D^n(ax+b)^n = m(m-1)(m-2)...(m--n+1)a^n(ax+b)^{m-n}$$

If m be a positive integer greater than n,
since $m(m-1)(m-2)...(m--n+1) = \frac{m!}{(m-n)!}$,
$\therefore D^n(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$.

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The nth derivatives of some special functions

$(iii)\ y = e^{ax}$

$\therefore\ y_1 = ae^{ax},\ y_2 = a^2e^{ax},\ y_3 = a^3e^{ax}, ...y_n = a^ne^{ax}.$

$\therefore\ \ D^n(e^{ax}) = a^ne^{ax}.$

$(iv)\ y = \frac{1}{a+x}$

$\therefore\ y_1 = -1(a+x)^{-2};\ \ y_2 = (-1)(-2)(a+x)^{-3} = (-1)^2 2!(a+x)^{-3}$

$$\therefore\ D^n\left(\frac{1}{a+x}\right) = \frac{(-1)^n n!}{(a+x)^{n+1}}$$

$(v)\ y = log(a+x)$

$$y_1 = \frac{1}{a+x}, \text{hence as in above}$$

$$D^n\{log(a+x)\} = \frac{(-1)^{n-1}(n-1)!}{(a+x)^n}$$

(vi)

$$
\begin{aligned}
y &= \sin(ax+b) \\
y_1 = a\cos(ax+b) &= a\sin(\frac{\pi}{2}+ax+b) \\
y_2 = a^2\cos(\frac{\pi}{2}+ax+b) &= a^2\sin(2\frac{\pi}{2}+ax+b) \\
y_3 = a^3\cos(2\frac{\pi}{2}+ax+b) &= a^3\sin(3\frac{\pi}{2}+ax+b) \\
\therefore\ D^n\{\sin(ax+b)\} &= a^n\sin(\frac{n\pi}{2}+ax+b) \\
Similarly\ \ D^n\{\cos(ax+b)\} &= a^n\cos(\frac{n\pi}{2}+ax+b)
\end{aligned}
$$

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Illustrative Example

Example 1: If $y = \sin^3$ **find** y_n

$$\begin{aligned}
\sin 3x &= 3\sin x - 4\sin^3 x \\
y &= \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x) \\
\therefore y_n &= \frac{1}{4}(3\sin(\frac{n\pi}{2} + x) - 3^n \sin 3(\frac{n\pi}{2} + x))
\end{aligned}$$

Example 2: If $y = \sin 3x \cos 2x$, **find** y_n

$$\begin{aligned}
y &= \sin 3x \cos 2x \\
&= \frac{1}{2}(\sin 5x + \sin x) \\
\therefore \quad y_n &= \frac{1}{2}(5^n \sin 5(\frac{n\pi}{2} + x) + \sin(\frac{n\pi}{2} + x))
\end{aligned}$$

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Illustrative Example

Example 3: If $y = e^{ax}\sin bx$, find y_n.

$$\begin{aligned} y &= e^{ax}\sin bx \\ y_1 &= e^{ax}a\sin bx + e^{ax}b\cos bx) \\ &= e^{ax}(a\sin bx + b\cos bx) \end{aligned}$$

Let $a = r\cos\phi, \quad b = r\sin\phi$, so that

$$\begin{aligned} r &= (a^2+b^2)^{\frac{1}{2}}, \quad \phi = \tan^{-1}\frac{b}{a} \\ \therefore\ y_1 &= re^{ax}\sin(bx+\phi) \end{aligned}$$

Similarly,

$$\begin{aligned} y_2 &= re^{ax}(a\sin(bx+\phi) + b\cos(bx+\phi)) \\ &= r^2e^{ax}\sin(bx+2\phi) \end{aligned}$$

In a similar way, $y_3 = r^3e^{ax}\sin(bx+3\phi)$ etc, and generally $y_n = r^ne^{ax}\sin(bx+n\phi)$.

Example 4: If $y = \frac{x^2+x-1}{x^3+x^2-6x}$ **find** x_n**.**

$x^3 + x^2 - 6x = x(x+3)(x-2)$

Let $\frac{x^2+x-1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$.

By using partial fraction we have found out

$A = \frac{1}{6}, \quad B = \frac{1}{3}, \quad C = \frac{1}{2}$.

$$\begin{aligned} \therefore \quad y &= \frac{1}{6}\frac{1}{x} + \frac{1}{3}\frac{1}{x+3} + \frac{1}{2}\frac{1}{x-2} \\ y_n &= (-1)^n n! \left\{ \frac{1}{6}.\frac{1}{x^{n+1}} + \frac{1}{3}.\frac{1}{(x+3)^{n+1}} + \frac{1}{2}.\frac{1}{(x-2)^{n+1}} \right\} \end{aligned}$$

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Illustrative Example

Exercise 5: If $y = \sin ax + \cos ax$**, show that** $y_n = a^n\{1 + (-1)^n \sin 2ax\}$**.**

Given,

$$
\begin{aligned}
y &= \sin ax + \cos ax \\
y_n &= a^n \sin\{\frac{n\pi}{2} + ax\} + b^n \cos\{\frac{n\pi}{2} + ax\} \\
&= a^n(\sin\{\frac{n\pi}{2} + ax\} + \cos\{\frac{n\pi}{2} + ax\})
\end{aligned}
$$

Since $(\sin ax + \cos ax) = (1 + \sin 2ax)^{\frac{1}{2}}$, then

$$
\begin{aligned}
y_n &= a^n\{1 + \sin 2(\frac{n\pi}{2} + ax)\} \\
&= a^n\{1 + \sin(\frac{2n\pi}{2} + 2ax)\}^{\frac{1}{2}} \\
&= \{1 + (-1)^n \sin 2ax\}^{\frac{1}{2}}
\end{aligned}
$$