NUMERICAL ANALYSIS: INTEGRATION (PAPER VIII GROUP B)- THIRD YEAR
Lecture 01

DR. AMAR NATH CHATTERJEE*

1 Numerical Integration

The subject matter of Numerical Integration is evaluate the definite integral $I = \int_a^b f(x)dx$.

The method of numerical integration or Quadrature are simple and a number of methods or rules have been developed. A quadrature formula is said to be closed type, if the limits of integration a and b are taken as interpolating points, otherwise the formula is called open type.

2 basic Concept of Quadrature

The definite integral $\int_a^b f(x)dx$ is interpreted as the area of the plane region bounded by the curve $y = f(x)$, the x axis and the two ordinates are a and b. The area may conveniently be evaluated by subdivision of the area into parts by division of the interval $[a, b]$ and then summation of the components areas. The additive property of the definite integral is explored to evaluate a definite integral in a piecewise seance. This is sometimes called the composite Rule.

3 Degree of Precision in a Quadrature formula

Let the values of the function $f(x)$ be known for a set of equispaced values of $x, a = x_o, x_1, x_2, ...x_n = b$. Also, let $f(x)$ be approximated an interpolation polynomial $\phi(x)$, such that $\phi(x_i) = f(x_i), i = 0, 1, 2, 3, ...$

Then $\int_a^b f(x)dx = \int_a^b \phi(x)dx + R$,

so that $r = \int_a^b f(x)dx - \int_a^b \phi(x)dx$, is known as the error of integration.

In this connection arises the idea of degree of precision.

*Email: anchaterji@gmail.com, Ph No. 7980114681
Google Classroom Code: csjrz73

4 A general Quadrature formula for equispaced arguments

As before, let the values of $y = f(x)$ corresponding to the values of equispaced arguments $a = x_0, x_1, x_2, ..., x_n = b$ be known to be $y_0, y_1, y_2, ..., y_n$ respectively.

Then Newton's Forward interpolation formula gives

$$\begin{aligned} y &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \\ &\quad ... + \frac{u(u-1)(u-2)...(u-\overline{n-1})}{n!}\Delta^n y_0, \end{aligned}$$

where $x_i = x_0 + ih,\ i = 1, 2, 3, ...$ and $x = x_0 + uh$.

Integrating noth side with respect to x, between the limits x_0 to x_n, we have

$$\begin{aligned} \int_{x_0}^{x_n} y dx &= \int_{x_0}^{x_n} [y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \\ &\quad ... + \frac{u(u-1)(u-2)...(u-\overline{n-1})}{n!}\Delta^n y_0] dx \\ &= h\int_0^n [y_0 + u\Delta y_0 + \frac{u^2-u}{2}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{6}\Delta^3 y_0 + ...] du. \end{aligned}$$

Since $x = x_0 + uh$, $u = 0$, when $x = x_0$ and $u = n$ when $x = x_n$, then

$$\int_a^b f(x)dx = nh[y_0 + \frac{n}{2}\Delta y_0 + \frac{2n^2-3n}{6}\Delta^2 y_0 + ...] \quad (1)$$

This is called general quadrature formula, known as Gauss-Legendre quadrature formula. from this, we can derive a number of integration formula by putting $n = 1, 2, 3, ...$

5 Trapezoidal Rule

We shall deduce the Trapezoidal Rule from the general quadrature formula.

putting n=1 in (1) and rejecting all differences above the first one, we have

$$\int_{x_0}^{x_1} = h[y_0 + \frac{1}{2}y_0]$$

The interval of integration being $[x_0, x_1]$, there are only two functions values y_0 and y_1, and with two values the only non zero difference is of the first order and higher order differences vanish.

$$\begin{aligned} \int_{x_0}^{x_1} &= h[y_0 + \frac{1}{2}(y_1 - y_0)], since \Delta y_0 = y_1 - y_0 \\ or, \quad \int_{x_0}^{x_1} ydx &= \frac{1}{2}[y_0 + y_1] \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{x_1}^{x_2} ydx &= \frac{h}{2}[y_1 + y_2] \\ \int_{x_2}^{x_3} ydx &= \frac{h}{2}[y_2 + y_3] \\ ... & \quad ...... \\ \int_{x_{n-1}}^{x_n} ydx &= \frac{h}{2}[y_{n-1} + y_n]. \end{aligned}$$

Adding all these

$$\begin{aligned} \int_{x_0}^{x_n} ydx &= \frac{h}{2}[(y_0 + y_1) + (y_1 + y_2) + ... + (y_{n-1} + y_n)] \\ &= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + +.. + y_{n-1})] \end{aligned}$$

This is called Trapezoidal Rule.