Lecture : 03

B. Sc. (Hon.) Part-I

Paper - II

# Partial Differential Equation

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# I. FORMATION OF PDE

## A. Formation of PDE by eliminating arbitrary constant

**1. Form a partial differential equation by the elimination of the constants h and k from** $(x-h)^{+}(y-k)^2+z^2=c^2$

$$(x-h)^{+}(y-k)^2+z^2 \;=\; c^2 \tag{1}$$

Differentiating partially w.r.t x and y we get,

$$2(x-h)+2zp \;=\; 0 \implies h=x+zp \tag{2}$$

$$2(y-k)+2zq \;=\; 0 \implies k=y+zq \tag{3}$$

Substituting $h,k$ from the original equation we obtain

$$z^2(p^2+q^2+1) \;=\; c^2 \tag{4}$$

**2. Form the partial differential equation corresponding to** $z=ax+by+ab$

$$z \;=\; ax+by+ab \tag{5}$$

Differentiating partially w.r.t x and y we get,

$$p \;=\; a;\; q=b. \tag{6}$$

Substituting a and b from the original equation we get

$$z \;=\; px+qy+pq. \tag{7}$$

**3. Eliminating a and b from** $z = a(x + y) + b$

$$z = a(x + y) + b \tag{8}$$

Differentiating partially w.r.t x and y we get,

$$p = a; \quad q = b \implies p = q. \tag{9}$$

**4. Eliminating a and b from** $z = ax + a^2y^2 + b$**.**

$$z = ax + a^2y^2 + b \tag{10}$$

Differentiating partially w.r.t x and y we get,

$$p = a; q = 2a^2y = 2p^2y \tag{11}$$

Therefore $q = 2p$ is the partial differential equation.

**5. Eliminating a and b from** $z = (x + a)(y + b)$

$$z = (x + a)(y + b) \tag{12}$$

Differentiating partially w.r.t x and y we get,

$$p = (y + b); q = (x + a) \implies z = pq \tag{13}$$

## II. ELIMINATION OF ARBITRARY FUNCTIONS

**1.Find the eliminate as a partial differential equation corresponding to the relation where f denotes an arbitrary function** $z = xf(\frac{y}{x})$

$$z = xf(\frac{y}{x}) \tag{14}$$

Let $v = \frac{y}{x}$ , then $v_x = -\frac{y}{x^2},\ v_y = \frac{1}{x}$. Differentiating the function partially w.r.t x

$$p \ = \ f(v) + xf'(v)vx = f(v) - \frac{y}{x}f'(v) \tag{15}$$

Again differentiating w.r.t y partially we get

$$q \ = \ xf'(v)v_y = f'(v) \tag{16}$$

Therefore $xp + yq = xf(v) = z$ Hence

$$z \ = \ xp + yq \tag{17}$$

This is the first order partially differential equation obtained by eliminating the arbitrary function f from the given relation.

**2. Find the PDE arrising from $\phi(x+y+z, x^2+y^2-z^2) = 0$ where $\phi$ denotes an arbitrary function**

The equation

$$\phi(x+y+z, x^2+y^2-z^2) \ = \ 0 \tag{18}$$

Let $x + y + z = u$ and $x^2 + y^2 - z^2 = v$ The equation form,

$$\phi(u, v) \ = \ 0 \tag{19}$$

Differentiating partially w.r.t x

$$\phi_u(u_x + pu_z) + \phi_v(v_x + pv_z) \ = \ 0 \tag{20}$$

$$\phi_u(u_y + qu_z) + \phi_v(v_y + qv_z) \ = \ 0 \tag{21}$$

Eliminating $\phi_u$ and $\phi_v$ we get

$$\begin{vmatrix} (\mathrm{u}_x + pu_z) & (\mathrm{v}_x + pv_z) \\ (\mathrm{u}_y + qu_z) & (\mathrm{v}_y + qv_z) \end{vmatrix} = 0$$

$$(1+p)(2y-2zq)-(2x-2zp)(1+q) \;=\; 0 \tag{22}$$
$$\implies y-x+z(p-q)+py-qx \;=\; 0 \tag{23}$$

which is a first order PDE by eliminating the arbitrary function $\phi$ from the given relation.

**3. Eliminating the arbitrary function** $f(y$ **and** $g(x)$ **from the relation** $z = xf(y)+yg(x)$

$$z \;=\; xf(y)+yg(x) \tag{24}$$

Differentiating partially w.r.t x and y we get

$$\frac{\partial z}{\partial x} \;=\; p = f(y)+yg'(y) \tag{25}$$
$$\frac{\partial z}{\partial y} \;=\; q = xf'(y)+g(x) \tag{26}$$

It is not possible to eliminate the arbitrary function $f, g, f', g'$ from equation $(1), (2), (3)$. We therefore consider second order derivatives Differentiating (2) partially w.r.t y on (3)

$$s \;=\; f'(y)+g'(y) \tag{27}$$

and differentiating (3) partially w.r.t x we get

$$\begin{aligned} s &= f'(y)+g'(x) \\ &= \frac{q-g(x)}{x}+\frac{p-f(y)}{y} \\ \implies xys &= xp+yq-z \end{aligned} \tag{28}$$