Lecture : 02

B. Sc. (Hon.) Part-I

Paper - II

Curvature

Dr. Amar Nath Chatterjee

Assistant professor

Department of Mathematics

K.L.S. College, Nawada

Email: anchaterji@gmail.com

I. FORMULA FOR RADIUS OF CURVATURE

A: For the cartesian equation $y = f(x)$

We know $\frac{dy}{dx} = \tan\phi$.

differentiatng with respect to x,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \sec^2\psi\frac{d\psi}{dx} = \sec^2\psi\frac{d\psi}{ds}.\frac{ds}{dx} \\ &= \sec^2\psi\frac{d\psi}{ds} \quad [\frac{dx}{ds} = \cos\psi] \\ \rho &= \frac{ds}{d\psi} = \sec^2\psi/\frac{d^2y}{dx^2}. \end{aligned}$$

Since $\sec\psi = (1+\tan^2\psi)^{\frac{1}{2}} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$.

$$\begin{aligned} \rho &= \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \\ &= \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \end{aligned}$$

B: For the Cartesian equation $x = f(y)$

We would found

$$\frac{dx}{dy} = \cot\psi, \quad \text{differentaiting with respect to y,}$$

$$\begin{aligned} \frac{d^2x}{d^2y} &= -{}^2\psi\frac{d\psi}{dy} = -{}^2\psi.\frac{d\psi}{dy}.\frac{ds}{dy} \\ &= -{}^2\psi.\frac{1}{\rho}.\ [\ \frac{dy}{ds} = \sin\psi] \end{aligned}$$

Since ${}^2\psi = 1+\cot^2\psi = 1+\left(\frac{dx}{dy}\right)^2$, considering the magnitude only of the radius of curvature

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}},$$

where $x_2 \neq 0$.

C: For the parametric equation $x = \phi(t), y = \psi(t)$

Here $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{y'}{x'}$, $(x' \neq 0)$, where dashes denote differentiations with respect to t.

$$Therefore\ \frac{d^2y}{dx^2} = \frac{d}{dx}\Big(\frac{y'}{x'}\Big) = \frac{d}{dt}\Big(\frac{y'}{x'}\Big).\frac{dt}{dx} = \frac{x'y''-y'x''}{x'^3}.$$

Then substituting the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ we get

$$\rho = \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-y'x''},$$

where dashes denote differentiations with respect to t, and where $x'y'' - x''y' \neq 0$.

D. For the Implicit equation $f(x, y) = 0$

Here,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}, (f_y \neq 0),\ i.e.\ f_x + f_y\frac{dy}{dx} = 0.$$

Differentiating this with respect to x,

$$f_{xx} + f_{xy}\frac{dy}{dx} + (f_{yx} + f_{yy}\frac{dy}{dx})\frac{dy}{dx} + f_y\frac{d^2y}{dx^2} = 0, or,\ f_{xx} + 2f_{xy}\frac{dy}{dx} + f_{yy}\Big(\frac{dy}{dx}\Big)^2 + f_y\frac{d^2y}{dx^2} = 0,$$

hence, replacing $-\frac{f_x}{f-y}$ for $\frac{dy}{dx}$ and simplifying,

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}$$

Substituting the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in the formula above, and considering the magnitude of ρ only, we get

$$\rho = \frac{(f_x^2 + f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

where $f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2 \neq 0$.

D: For the polar equation $r = f(\theta)$.

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}.\frac{d\theta}{d\psi} = \frac{ds}{d\theta}\Big/\frac{d\psi}{d\theta}.$$

Now, $\psi = \theta + \phi = \theta + \tan^{-1}\frac{r}{r_1}$ where, $r_1 = \frac{dr}{d\theta}$. Therefore

$$\frac{d\psi}{d\theta} = \frac{r^2 + 2r_1^2 - rr_2}{r^2 + r_1^2}.$$

Again, $\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}$ From above we get

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}$$